

**SEMINAR ON “TILTING MODULES AND TILTING TORSION
THEORIES” WRITTEN BY R. COLPI AND J. TRLIFAJ**

MAURICIO MEDINA-BÁRCENAS

ABSTRACT. These notes were made during a graduate seminar at Benemérita Universidad Autónoma de Puebla in Spring-2020. In the seminar, we studied the paper [4].

1. PRELIMINARIES

It is assumed that the reader is familiar with the left and right derived functors. The functor Ext will play a central roll in this notes. The general background on derived functors can be found in [13]. For convenience of the reader we will mention some results which will be used along the manuscript.

Proposition 1.1. (1) *If $\{A_k\}_{k \in K}$ is a family of modules, then there are natural isomorphisms, for all $n > 0$,*

$$\text{Ext}_R^n \left(\bigoplus_{k \in K} A_k, B \right) \cong \prod_{k \in K} \text{Ext}_R^n(A_k, B).$$

(2) *If $\{B_k\}_{k \in K}$ is a family of modules, then there are natural isomorphisms, for all $n > 0$,*

$$\text{Ext}_R^n \left(A, \prod_{k \in K} B_k \right) \cong \prod_{k \in K} \text{Ext}_R^n(A, B_k).$$

Proof. [13, Proposition 7.21 and 7.22]. □

Proposition 1.2. (1) *A left R -module P is projective if and only if $\text{Ext}_R^n(P, B) = 0$ for every R -module B .*

(2) *A left R -module E is injective if and only if $\text{Ext}_R^n(A, E) = 0$ for every R -module A .*

Proof. (1) [13, Corollary 6.58 and Corollary 7.25].

(2) [13, Corollary 6.41 and Corollary 7.25]. □

Definition 1.3. Let A be a left R -module. The *projective dimension* of A is a less or equal than n ($pd(A) \leq n$), if there is a finite projective resolution

$$0 \rightarrow P_n \rightarrow \cdots \rightarrow P_1 \rightarrow P_0 \rightarrow A \rightarrow 0.$$

If no such finite resolution exists, then $pd(A) = \infty$; otherwise, $pd(A) = n$ if n is the length of a shortest projective resolution of A .

Proposition 1.4. *The following are equivalent for a left R -module A .*

(a) $pd(A) \leq n$

(b) $\text{Ext}_R^k(A, B) = 0$ for all left R -modules B and all $k \geq n + 1$.

Proof. [13, Proposition 8.6]. \square

Definition 1.5. A ring R is said to be *left hereditary* if every left ideal is projective.

Proposition 1.6. *The following conditions are equivalent for a ring R :*

- (a) R is left hereditary;
- (b) Submodules of projective left R -modules are projective;
- (c) Factor modules of injective left R -modules are injective.

Corollary 1.7. *Let R be a left hereditary ring. Then $\text{pd}(A) \leq 1$ for all left R -module A .*

Given two modules A and C , an extension of A by C is an exact sequence $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$. The next Proposition shows that $\text{Ext}^1(C, A)$ detects the nontrivial extensions of A by C .

Proposition 1.8. *Every extension $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ splits if and only if $\text{Ext}^1(C, A) = 0$.*

Proof. [13, Proposition 7.24 and Theorem 7.31]. \square

I want to show the general idea of how from an extension of A by C we get an element in $\text{Ext}^1(C, A)$ and vice-versa, all the details can be found in [13, Ch. 7, Sec. 2]. To get this we will need the constructions of pullback and pushout in modules. Let us start with $[\alpha] \in \text{Ext}^1(C, A)$. Taking an injective resolution

$$\mathbf{E} \quad 0 \longrightarrow A \xrightarrow{\eta} E_0 \xrightarrow{d_0} E_{-1} \xrightarrow{d_{-1}} E_{-2} \longrightarrow$$

of A and applying the functor $\text{Hom}(C, -)$ to the reduced resolution \mathbf{E}_A we get the complex:

$$\text{Hom}(C, \mathbf{E}_A) \quad 0 \longrightarrow \text{Hom}(C, E_0) \xrightarrow{(C, d_0)} \text{Hom}(C, E_{-1}) \xrightarrow{(C, d_{-1})}$$

Then $\text{Ext}^1(C, A) := H^1(\text{Hom}(C, \mathbf{E}_A)) = \text{Ker}(C, d_{-1}) / \text{Im}(C, d_0)$. This implies that $[\alpha] \in \text{Ext}^1(C, A)$ is represented by a morphism $\alpha : C \rightarrow E_{-1}$ such that $d_{-1}\alpha = 0$. Hence $\alpha(C) \subseteq \text{Ker } d_{-1} = \text{Im } d_0$. Therefore, there is a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & A & \longrightarrow & M & \dashrightarrow & C & \longrightarrow & 0 \\ & & \downarrow \mathbf{1} & & \downarrow & & \downarrow \alpha & & \\ 0 & \longrightarrow & A & \xrightarrow{\eta} & E_0 & \xrightarrow{d_0} & \text{Im } d_0 & \longrightarrow & 0 \end{array}$$

where M is the pullback of the angle given by α and d_0 . Thus, we have an extension of A by C .

Remark 1.9. The construction that we just made, can be done using a projective resolution of C . In this case, $\alpha : P_1 \rightarrow A$ and the extension is given by a pushout (see [13, Theorem 7.30]).

Conversely, suppose that we have an extension $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ of A by C . Consider the injective resolution \mathbf{E} of A . Then we have a morphism of complexes

over the identity 1_A :

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\
& & \downarrow 1 & & \downarrow \alpha_0 & & \downarrow \alpha_1 & & \downarrow \alpha_2 \\
0 & \longrightarrow & A & \xrightarrow{\eta} & E_0 & \xrightarrow{d_0} & E_{-1} & \xrightarrow{d_{-1}} & E_{-2}
\end{array}$$

Hence $d_{-1}\alpha_1 = \alpha_2 0 = 0$. This implies that $\alpha_1 \in \text{Ker}(C, d_{-1})$. Thus, $[\alpha_1] \in \text{Ext}^1(C, A)$.

Lemma 1.10. *Let $0 \longrightarrow A \xrightarrow{i} B \xrightarrow{\rho} C \longrightarrow 0$ be an exact sequence and let M be a module. Then the connection morphism $\partial : \text{Hom}(M, C) \rightarrow \text{Ext}^1(M, A)$ is given by taking the pullback along ρ . That is, given $f \in \text{Hom}(M, A)$, $\partial(f)$ corresponds to the extension:*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \xrightarrow{j} & L & \xrightarrow{\pi} & M & \longrightarrow & 0 \\
& & \downarrow 1 & & \downarrow \gamma & & \downarrow -f & & \\
0 & \longrightarrow & A & \xrightarrow{i} & B & \xrightarrow{\rho} & C & \longrightarrow & 0
\end{array}$$

Proof. We have to recall how the connection morphism was defined. Let us take injective resolutions

$$\begin{array}{l}
E' \quad 0 \longrightarrow A \xrightarrow{\eta'} E'_0 \xrightarrow{d'_0} E'_{-1} \xrightarrow{d'_{-1}} E'_{-2} \longrightarrow \\
E'' \quad 0 \longrightarrow C \xrightarrow{\eta''} E''_0 \xrightarrow{d''_0} E''_{-1} \xrightarrow{d''_{-1}} E''_{-2} \longrightarrow
\end{array}$$

of A by C respectively. Using the dual version of the horseshoe lemma, we have an injective resolution

$$E \quad 0 \longrightarrow B \xrightarrow{\eta} E_0 \xrightarrow{d_0} E_{-1} \xrightarrow{d_{-1}} E_{-2} \longrightarrow$$

of B such that $E_j = E'_j \oplus E''_j$. Therefore, there is a commutative diagram:

$$\begin{array}{ccccccccc}
& & 0 & & 0 & & 0 & & \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A & \xrightarrow{i} & B & \xrightarrow{\rho} & C & \longrightarrow & 0 \\
& & \downarrow \eta' & \nearrow \sigma_0 & \downarrow \eta & & \downarrow \eta'' & & \\
0 & \longrightarrow & E'_0 & \xrightarrow{\zeta_0} & E_0 & \xrightarrow{\xi_0} & E''_0 & \longrightarrow & 0 \\
& & \downarrow d'_0 & & \downarrow d_0 & & \downarrow d''_0 & & \\
0 & \longrightarrow & E'_{-1} & \xrightarrow{\zeta_{-1}} & E_{-1} & \xrightarrow{\xi_{-1}} & E''_{-1} & \longrightarrow & 0 \\
& & \downarrow d'_{-1} & & \downarrow d_{-1} & & \downarrow d''_{-1} & & \\
0 & \longrightarrow & E'_{-2} & \xrightarrow{\zeta_{-2}} & E_{-2} & \xrightarrow{\xi_{-2}} & E''_{-2} & \longrightarrow & 0
\end{array}$$

Applying the functor to the reduced resolutions, we get

$$\begin{array}{ccccccc}
0 & \longrightarrow & \text{Hom}(M, E'_0) & \xrightarrow{(M, \zeta_0)} & \text{Hom}(M, E_0) & \xrightarrow{(M, \xi_0)} & \text{Hom}(M, E''_0) \longrightarrow 0 \\
& & \downarrow (M, d'_0) & & \downarrow (M, d_0) & & \downarrow (M, d''_0) \\
0 & \longrightarrow & \text{Hom}(M, E'_{-1}) & \xrightarrow{(M, \zeta_{-1})} & \text{Hom}(M, E_{-1}) & \xrightarrow{(M, \xi_{-1})} & \text{Hom}(M, E''_{-1}) \longrightarrow 0
\end{array}$$

Moreover, $\text{Ker}(M, d''_0) \cong \text{Hom}(M, C)$, where the isomorphism is given by $\eta'' \circ _$. Now, $\eta : B \rightarrow E_0$ is defined as $\eta(b) = (\sigma_0(b), \eta'' \rho(b))$ where $\sigma_0 : B \rightarrow E'_0$ is such that $\sigma_0 i = \eta'$. Analogously, $d_0 : E_0 \rightarrow E_{-1}$ is defined as $d_0(x) = (\sigma_1(\bar{x}), d''_0 \xi_0(x))$ where $\sigma_1 : \text{Coker } \eta \rightarrow E'_{-1}$. Then $\partial(f) = [\alpha] \in \text{Ext}^1(M, A)$ is given by the morphism $\alpha \in \text{Hom}(M, E'_{-1})$ such that $\alpha = \widehat{\zeta_{-1}} d_0 \xi_0 \eta'' f$ where $\xi_0 : E'_0 \rightarrow E_0$ is defined as $\xi_0(x) = (0, x)$ and $\widehat{\zeta_{-1}} : E_{-1} \rightarrow E'_{-1}$ is defined as $\widehat{\zeta_{-1}}(x, y) = x$. Set $\beta = \widehat{\zeta_{-1}} d_0 \xi_0 \eta''$. Note that

$$\begin{aligned}
\beta \rho(b) &= \widehat{\zeta_{-1}} d_0(0, \eta'' \rho(b)) = \widehat{\zeta_{-1}}(\sigma_1(\overline{0, \eta'' \rho(b)}), d''_0 \eta'' \rho(b)) \\
&= \widehat{\zeta_{-1}}(\sigma_1(\overline{0, \eta'' \rho(b)}), 0) = \sigma_1(\overline{0, \eta'' \rho(b)}).
\end{aligned}$$

On the other hand, $d'_0 \sigma_0(b) = \zeta_{-1} \sigma_1(\overline{\zeta_0 \sigma_0(b)}) = \sigma_1(\overline{\sigma_0(b), 0})$. Thus, we have the following diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & A & \xrightarrow{j} & L & \xrightarrow{\pi} & M \longrightarrow 0 \\
& & \downarrow 1 & & \downarrow \sigma_0 \gamma & & \downarrow \alpha \\
0 & \longrightarrow & A & \xrightarrow{\eta'} & E'_0 & \xrightarrow{d'_0} & \text{Im } d'_0 \longrightarrow 0.
\end{array}$$

Let us see that this diagram is commutative. Let $a \in A$. Then $\sigma_0 \gamma j(a) = \sigma_0 \gamma(i(a), 0) = \sigma_0(i(a)) = \sigma_0(i(a)) = \eta'(a) = \eta'(1(a))$. On the other hand, $d'_0 \sigma_0 \gamma(b, m) = d'_0 \sigma_0(b) = \sigma_1(\overline{\sigma_0(b), 0}) = -\sigma_1(\overline{0, \eta'' \rho(b)}) = -\beta \rho(b)$. Since $(b, m) \in L$, $\rho(b) = -f(m)$. Then $d'_0 \sigma_0 \gamma(b, m) = -\beta \rho(b) = \beta f(m) = \alpha \pi(b, m)$. This implies that $\partial(f) = [\alpha] \in \text{Ext}^1(M, A)$ is the element which corresponds to the extension

$$0 \longrightarrow A \xrightarrow{j} L \xrightarrow{\pi} M \longrightarrow 0$$

□

Definition 1.11. Let M be an R -module. The *Ext-orthogonal class* of M is given by

$$M^\perp = \{ {}_R N \mid \text{Ext}^1_R(M, N) = 0 \}.$$

Definition 1.12. Given two R -modules M and N , it is said that N is M -generated if there exists an epimorphism $\rho : M^{(X)} \rightarrow N$ for some set X . The class of modules M -generated is denoted by $\text{Gen}(M)$.

It is not difficult to see that the class M^\perp is closed under extensions and contains all the injective R -modules by Proposition 1.2, and the class $\text{Gen}(M)$ is closed under direct sums and epimorphisms, for all modules M .

Definition 1.13. A module M is *finendo* if M is finitely generated as module over its endomorphism ring.

Lemma 1.14 (Lemma 1.5, [3]). *Let M be a module. Then, $M^X \in \text{Gen}(M)$ for every set X if and only if M is finendo.*

Proof. \Rightarrow Suppose that $M^M \in \text{Gen}({}_R M)$. Then, there exists an R -epimorphism $\rho : M^{(I)} \rightarrow M^M$ for some set I . Let $(x_m)_{m \in M}$ be the element in M^M such that $x_m = m$. Hence, there exists $(y_x)_{x \in I} \in M^{(I)}$ such that $\rho((y_x)_{x \in I}) = (x_m)_{m \in M}$. This implies that there is a finite subset $F \subseteq I$ and homomorphisms $\rho_i : M \rightarrow M^M$ such that $(x_m)_{m \in M} = \rho((y_x)_{x \in I}) = \sum_{i \in F} \rho_i(y_i)$. For each $m \in M$, let π_m denote the canonical projection $\pi_m : M^M \rightarrow M$ and set $f_i^m = \pi_m \rho_i \in \text{End}_R(M)$. Therefore,

$$m = \pi_m((x_m)_{m \in M}) = \pi_m \left(\sum_{i \in F} \rho_i(y_i) \right) = \sum_{i \in F} (\pi_m \rho_i(y_i)) = \sum_{i \in F} f_i^m(y_i),$$

for each $m \in M$. Thus, M as module over $\text{End}_R(M)$ is generated by $\{y_i \mid i \in F\}$.

\Leftarrow Let $S = \text{End}_R(M)$ and suppose ${}_S M$ is generated by $\{y_1, \dots, y_n\}$. Let X be any set and $(m_x)_{x \in X} \in M^X$. For each $x \in X$ there exist $f_1^x, \dots, f_n^x \in S$ such that $m_x = \sum_{i=1}^n f_i^x(y_i)$. Let $\phi_x : M^n \rightarrow M$ be the homomorphism given by $\phi_x(m_1, \dots, m_n) = \sum_{i=1}^n f_i^x(m_i)$ and let $\phi : M^n \rightarrow M^X$ be the homomorphism given by $\phi(m_1, \dots, m_n) = (\phi_x(m_1, \dots, m_n))_{x \in X}$. Consider the element $(y_1, \dots, y_n) \in M^n$. Then,

$$\phi(y_1, \dots, y_n) = (\phi_x(y_1, \dots, y_n))_{x \in X} = \left(\sum_{i=1}^n f_i^x(y_i) \right)_{x \in X} = (m_x)_{x \in X}.$$

This implies that $(m_x)_{x \in X} \in \text{tr}^M(M^X)$. Thus, M^X is M -generated. \square

Lemma 1.15. *Let T be an R -module. If $\text{Gen}(T) = T^\perp$, then $\text{Gen}(T) = \text{Pres}(T)$.*

Proof. Let $M \in \text{Gen}(T)$ and $X = \text{Hom}_R(T, M)$. Then, there is an exact sequence $0 \rightarrow K \rightarrow T^{(X)} \xrightarrow{\rho} M \rightarrow 0$. Applying the functor $\text{Hom}_R(T, -)$, we get

$$\longrightarrow \text{Hom}_R(T, T^{(X)}) \xrightarrow{\rho_*} \text{Hom}_R(T, M) \longrightarrow \text{Ext}^1(T, K) \longrightarrow 0$$

Note that $\text{Ext}^1(T, T^{(X)}) = 0$, by hypothesis. We claim that ρ_* is surjective. For, consider $h \in \text{Hom}_R(T, M)$. Let $\eta_h : T \rightarrow T^{(X)}$ be the canonical inclusion. Then, $\rho_*(\eta_h)(t) = \rho \eta_h(t) = h(t)$. Thus ρ_* is surjective and hence $\text{Ext}^1(T, K) = 0$. This implies that $K \in T^\perp = \text{Gen}(T)$. So, $\text{Gen}(T) \subseteq \text{Pres}(T)$. We always have that $\text{Pres}(T) \subseteq \text{Gen}(T)$. \square

Lemma 1.16. *Let M and N be R -modules such that $N \in \text{Pres}(M)$ and $\text{Gen}(M) \subseteq N^\perp$. Then, $N \in \text{Add}(M)$.*

Proof. By hypothesis, there is an exact sequence $0 \rightarrow K \rightarrow M^{(X)} \rightarrow N \rightarrow 0$ with $K \in \text{Gen}(M)$. Since $\text{Gen}(M) \subseteq N^\perp$, $\text{Ext}^1(N, K) = 0$. This implies that the sequence splits. Thus, $N \in \text{Add}(M)$. \square

Lemma 1.17. *Let M be a left R -module. Then, M^\perp is closed under epimorphisms if and only if $\text{pd}(M) \leq 1$.*

Proof. \Rightarrow There exists an exact sequence $0 \rightarrow K \rightarrow R^{(X)} \rightarrow M \rightarrow 0$ for some set X . We claim that K is projective. Let N be any module. By Proposition 1.2, $\text{Ext}^1(R^{(X)}, N) = 0 = \text{Ext}^2(R^{(X)}, N)$. It follows that there is an exact sequence $0 \rightarrow \text{Ext}^1(K, N) \rightarrow \text{Ext}^2(M, N) \rightarrow 0$, and so $\text{Ext}^1(K, N) \cong \text{Ext}^2(M, N)$. Let $E(N)$ denote the injective hull of N and consider the exact sequence $0 \rightarrow N \rightarrow E(N) \rightarrow E(N)/N \rightarrow 0$. Again by Proposition 1.2, $\text{Ext}^1(M, E(N)) = 0 = \text{Ext}^2(M, E(N))$.

Therefore $\text{Ext}^1(M, E(N)/N) \cong \text{Ext}^2(M, N)$. Hence $\text{Ext}^1(M, E(N)/N) \cong \text{Ext}^1(K, N)$. Since $E(N) \in M^\perp$, $E(N)/N \in M^\perp$ by hypothesis. Thus, $\text{Ext}^1(K, N) \cong \text{Ext}^1(M, E(N)/N) = 0$. Since N was an arbitrary module, it follows that K is projective. Thus, $\text{pd}(M) \leq 1$.

\Leftarrow Let $N \in M^\perp$ and let $\rho : N \rightarrow L$ be an epimorphism. Set $K = \text{Ker } \rho$. There is an exact sequence $0 \rightarrow K \rightarrow N \xrightarrow{\rho} L \rightarrow 0$. Applying the functor $\text{Hom}_R(M, -)$ to that sequence, we get an exact sequence

$$\cdots \rightarrow \text{Ext}^1(M, N) \rightarrow \text{Ext}^1(M, L) \rightarrow \text{Ext}^2(M, K) \rightarrow \cdots$$

It follows that $\text{Ext}^2(M, K) = 0$ by Proposition 1.4 and $\text{Ext}^1(M, N) = 0$ because $N \in M^\perp$. Therefore, $\text{Ext}^1(M, L) = 0$. Thus, $L \in M^\perp$. \square

Corollary 1.18. *Let T be a module. Then $\text{pd}(M) \leq 1$ and $\text{Ext}^1(T, T^{(X)}) = 0$ for any set X if and only if $\text{Gen}(T) \subseteq T^\perp$ and T^\perp is closed under epimorphisms.*

Proof. By Lemma 1.17, T^\perp is closed under epimorphisms if and only if $\text{pd}(T) \leq 1$. For any set X , $\text{Ext}^1(T, T^{(X)}) = 0$, that is $T^{(X)} \in T^\perp$. Therefore, $\text{Gen}(T) \subseteq T^\perp$. Reciprocally, if $\text{Gen}(T) \subseteq T^\perp$, $\text{Ext}^1(T, T^{(X)}) = 0$. \square

Definition 1.19. Let M be a left R -module. The module M is called *small* if the functor $\text{Hom}_R(M, -)$ commutes with direct sums canonically.

Remark 1.20. Every finitely generated module is small

Proposition 1.21 (Lemma 1.2, [14]). *The following conditions are equivalent for a module M .*

- (a) M is small and M^\perp is closed under direct sums and epimorphisms;
- (b) M is finitely generated and $\text{pd}(M) \leq 1$.

Proof. There exists an exact sequence

$$(1.1) \quad 0 \rightarrow K \rightarrow R^{(X)} \rightarrow M \rightarrow 0$$

for some set X . By Lemma 1.17, K is projective. It follows from [1] that $K = \bigoplus_{\alpha \in \Lambda} K_\alpha$ is a direct sum of countable generated modules K_α . Set $D = \bigoplus_{\alpha \in \Lambda} E(K_\alpha)$. There is a canonical inclusion $K \hookrightarrow D$. Since M^\perp is closed under direct sums, $\text{Ext}^1(M, D) = 0$. Applying $\text{Hom}_R(-, D)$ to the sequence (1.1), we get an epimorphism

$$\cdots \rightarrow \text{Hom}_R(R^{(X)}, D) \rightarrow \text{Hom}_R(K, D) \rightarrow 0.$$

Therefore, there exists $g : R^{(X)} \rightarrow D$ such that $g|_K = i$.

$$\begin{array}{ccccc} 0 & \longrightarrow & K & \longrightarrow & R^{(X)} \\ & & \downarrow i & \swarrow g & \\ & & D & & \end{array}$$

For each $\alpha \in \Lambda$, let $\pi_\alpha : D \rightarrow E(K_\alpha)$ and $\rho_\alpha : E(K_\alpha) \rightarrow E(K_\alpha)/K_\alpha$ be the canonical projections, respectively, and set $g_\alpha = \rho_\alpha \pi_\alpha g$, that is, the composition

$$R^{(X)} \xrightarrow{g} \bigoplus_{\alpha \in \Lambda} E(K_\alpha) \xrightarrow{\pi_\alpha} E(K_\alpha) \xrightarrow{\rho_\alpha} E(K_\alpha)/K_\alpha$$

Now, define $h : R^{(X)} \rightarrow \bigoplus_{\alpha \in \Lambda} E(K_\alpha)/K_\alpha$ as $h(x) = (g_\alpha(x))_{\alpha \in \Lambda}$. Note that $g_\alpha(K) = 0$ for each α , hence $K \leq \text{Ker } h$. From (1.1), $M \cong R^{(X)}/K$. Therefore, h induces a homomorphism $\bar{h} \in \text{Hom}_R(M, \bigoplus_{\alpha \in \Lambda} E(K_\alpha)/K_\alpha)$. Since M is small,

there exists a finite subset $F \subseteq \Lambda$ such that $\text{Im } \bar{h} \subseteq \bigoplus_{\alpha \in F} E(K_\alpha)/K_\alpha$. Thus, $\text{Im } g \subseteq \bigoplus_{\alpha \in F} E(K_\alpha) + \bigoplus_{\alpha \in \Lambda} K_\alpha$. Let π denote the projection of D onto $\bigoplus_{\alpha \notin F} E(K_\alpha)$ and let \bar{K} denote $\bigoplus_{\alpha \notin F} K_\alpha$. If $\bar{g} = \pi g$, then $\bar{g} \in \text{Hom}_R(R^{(X)}, \bar{K})$. Since $\bar{g}|_{\bar{K}} = id$, $R^{(X)} = \text{Ker } \bar{g} \oplus \bar{K}$. Set $A = \text{Ker } \bar{g} \cap \bar{K} = \bigoplus_{\alpha \in F} K_\alpha$. It follows that

$$M = R^{(X)}/K = \frac{\text{Ker } \bar{g} + K}{K} \cong \text{Ker } \bar{g}/A.$$

Since $\text{Ker } \bar{g}$ is projective, by [1], $\text{Ker } \bar{g} = \bigoplus_{\beta \in \Xi} C_\beta$ is a direct sum of countable generated modules. Also, A is a direct sum of countable generated modules. Hence M is direct sum of a countable generated module C and a projective module B ,

$$M \cong \text{Ker } \bar{g}/A = \bigoplus_{\beta \in \Xi} C_\beta / \bigoplus_{\alpha \in F} K_\alpha \cong C \oplus B.$$

Since M is small, B is countable generated. Thus, M is a small countable generated module. Therefore M is finitely generated, by [14, Lemma 1.1].

\Leftarrow By Lemma 1.17, M^\perp is closed under epimorphisms. Since M is finitely generated, M is small. Now, let $\{B_i\}_{i \in I}$ be a family of modules in M^\perp . It is not difficult to see that, if M is small then $\text{Ext}^n(M, _)$ commutes with direct sums for all $n > 0$. Hence,

$$\text{Ext}^1 \left(M, \bigoplus_{i \in I} B_i \right) \cong \bigoplus_{i \in I} \text{Ext}^1(M, B_i)$$

Since each $B_i \in M^\perp$, $\text{Ext}^1(M, B_i) = 0$. Therefore $\text{Ext}^1(M, \bigoplus_{i \in I} B_i) = 0$. Thus, $\bigoplus_{i \in I} B_i \in M^\perp$. \square

Corollary 1.22. *If M satisfies any of the conditions in Proposition 1.21, then M is finitely presented.*

Proof. Since M is finitely generated and $pd(M) \leq 1$, there is an exact sequence

$$(1.2) \quad 0 \rightarrow P \rightarrow R^{(n)} \rightarrow M \rightarrow 0$$

with P projective. Since P is projective, P is a direct summand of a free module $R^{(Y)}$. Let $j : P \rightarrow R^{(Y)}$ be the canonical inclusion, let $g : R^{(Y)} \rightarrow E(R)^{(Y)}$ be the canonical monomorphism and set $f = gj$. Since M^\perp is closed under direct sums, $E(R)^{(Y)} \in M^\perp$ and so $\text{Ext}^1(M, E(R)^{(Y)}) = 0$. Thus, there is an exact sequence

$$0 \rightarrow \text{Hom}_R(M, E(R)^{(Y)}) \rightarrow \text{Hom}_R(R^{(n)}, E(R)^{(Y)}) \rightarrow \text{Hom}_R(P, E(R)^{(Y)}) \rightarrow 0.$$

This implies that f can be extended to a homomorphism $\hat{f} : R^{(n)} \rightarrow E(R)^{(Y)}$. Let $\pi_y : E(R)^{(Y)} \rightarrow E(R)$ and $\rho_y : R^{(Y)} \rightarrow R$ be the canonical projections for each $y \in Y$. Since $R^{(n)}$ is fin. gen. $F = \{y \in Y \mid \pi_y \hat{f} \neq 0\}$ is finite. It follows that ρ_y is the corestriction of $\pi_y g$ to R . Let $y \in Y \setminus F$. Then $\pi_y \hat{f} = 0$ and so $\pi_y f = 0$. This implies that $\rho_y j = \pi_y g j = \pi_y f = 0$. Hence $P = \text{Im } j \subseteq R^{(F)}$ and so P is a direct summand of a fin. gen. free module. Thus, P is fin. gen. and M is finitely presented. \square

Lemma 1.23. *Let M be an R -module. If M is faithful and finendo, then*

- (1) *there exists an exact sequence $0 \rightarrow R \xrightarrow{i} M^n \rightarrow M' \rightarrow 0$ for some $n > 0$.*
- (2) *for any module L , the induced homomorphism $i^* : \text{Hom}_R(M, L) \rightarrow \text{Hom}_R(R, L)$ is surjective if and only if $L \in \text{Gen}(M)$.*
- (3) $M'^\perp \subseteq \text{Gen}(M)$.

(4) M generates every injective module.

Proof. (1) Set $S = \text{End}_R(M)$ and let $\{t_1, \dots, t_n\}$ be a set of generators of ${}_S M$. Since M is faithful, there is a monomorphism $i : R \rightarrow M^n$ given by $i(r) = (rt_1, \dots, rt_n)$. Hence, we have the exact sequence

$$0 \rightarrow R \xrightarrow{i} M^n \rightarrow M^n/R \rightarrow 0.$$

(2) \Rightarrow Suppose $i^* : \text{Hom}_R(M^n, L) \rightarrow \text{Hom}_R(R, L)$ is surjective and let $l \in L$. Since $\text{Hom}_R(R, L) \cong L$, there exists $g \in \text{Hom}_R(M^n, L)$ such that $gi(1) = l$. Thus, $l \in \text{tr}^M(L)$ and so $L \in \text{Gen}(M)$.

\Leftarrow Applying $\text{Hom}_R(-, L)$ to the exact sequence, we get

$$0 \rightarrow \text{Hom}_R(M', L) \rightarrow \text{Hom}_R(M^n, L) \xrightarrow{i^*} \text{Hom}_R(R, L)$$

Let $x \in L \cong \text{Hom}_R(R, L)$. Since $L \in \text{Gen}(M)$, there is an epimorphism $M^{(Y)} \rightarrow L$ for some set Y . Making the inverse image of x , we have a homomorphism $f : M^m \rightarrow L$ and $(x_1, \dots, x_m) \in M^m$ such that $f(x_1, \dots, x_m) = x$. Since ${}_S M = \langle t_1, \dots, t_n \rangle$, for each $1 \leq i \leq m$ there exists $f_1^i, \dots, f_n^i \in S$ such that $x_i = \sum_{j=1}^n f_j^i(t_j)$. Define $\alpha : M^n \rightarrow M^m$ as $\alpha(y_1, \dots, y_n) = \left(\sum_{j=1}^n f_j^1(y_j), \dots, \sum_{j=1}^n f_j^m(y_j) \right)$. Therefore, $\alpha(t_1, \dots, t_n) = (x_1, \dots, x_m)$. This implies that $i^*(f\alpha)(1) = f\alpha i(1) = f(x_1, \dots, x_m) = x$. Proving that i^* is surjective.

(3) If $L \in M'^{\perp}$, i.e. $\text{Ext}^1(M', L) = 0$, then i^* is surjective. Thus, $M \in \text{Gen}(P)$.

(4) By (1), there is a monomorphism $i : R \rightarrow M^n$. Let E be an injective module and $\phi : R^{(X)} \rightarrow E$ be an epimorphism. Then ϕ can be extended to an epimorphism $\bar{\phi} : (M^n)^{(X)} \rightarrow E$. Thus, $E \in \text{Gen}(M)$. \square

2. TILTING MODULES

Definition 2.1. A left R -module T is *tilting* if satisfies the following conditions:

- (T1) There is an exact sequence $0 \rightarrow R \rightarrow T' \rightarrow T'' \rightarrow 0$ such that $T', T'' \in \text{Add}(T)$.
- (T2) $\text{Ext}^1(T, T^{(X)}) = 0$ for any set X .
- (T3) $\text{pd}(T) \leq 1$.

If, moreover, T satisfies the condition

- (T4) T is finitely presented,

then T is a *classical tilting module*. A module T is a *classical partial tilting module* provided (T2),(T3) and (T4) hold true.

- Proposition 2.2.**
- (1) A left R -module T is tilting if and only if $\text{Gen}(T) = T^{\perp}$.
 - (2) A left R -module P is classical partial tilting if and only if P is small, $\text{Gen}(P) \subseteq P^{\perp}$ and P^{\perp} is a torsion class.
 - (3) A left R -module T is classical tilting if and only if T is (self-) small and $\text{Gen}(T) = T^{\perp}$.

Proof. (1) \Rightarrow Suppose, T is a tilting module. Then, $\text{Gen}(T) \subseteq T^{\perp}$ by Corollary 1.18. Now, by (T1), there is an exact sequence $0 \rightarrow R \xrightarrow{\alpha} T' \rightarrow T'' \rightarrow 0$ with $T', T'' \in \text{Add}(T)$. Let $M \in T^{\perp}$. Then $\text{Ext}^1(T'', M) = 0$ (see Proposition 1.1). Hence, there is an exact sequence

$$0 \longrightarrow \text{Hom}_R(T'', M) \longrightarrow \text{Hom}_R(T', M) \xrightarrow{\alpha^*} \text{Hom}_R(R, M) \longrightarrow 0.$$

Since $\text{Hom}_R(R, M) \cong M$, this implies that for each $m \in M$, there exists $g \in \text{Hom}_R(T', M)$ such that $g(\alpha(1)) = m$. Therefore, M is T -generated.

\Leftarrow Since $\text{Gen}(T) = T^\perp$, T^\perp is closed under direct sums and epimorphisms. This implies that $\text{Ext}^1(T, T^{(X)}) = 0$ for any set X and $pd(M) \leq 1$ by Lemma 1.17. Since $E(R) \in \text{Gen}(T)$, there is an epimorphism $\rho : T^{(X)} \rightarrow E(R)$ for some set X . Consider $i : R \rightarrow E(R)$ the canonical inclusion, then there exists a monomorphism $j : R \rightarrow T^{(X)}$ such that $\rho j = i$. This implies that T is faithful, i.e., $\text{Ann}(T) = 0$. On the other hand, since $\text{Ext}^1(T, T) = 0$, by Proposition 1.1, $\text{Ext}^1(T, T^Y) = 0$ for any set Y . That is $T^Y \in T^\perp = \text{Gen}(T)$ for any set Y , thus T is finendo by Lemma 1.14. Set $S = \text{End}_R(T)$ and let $\{t_1, \dots, t_n\}$ be a set of generators of ${}_S T$. By Lemma 1.23.(1), there is an exact sequence

$$(2.1) \quad 0 \rightarrow R \xrightarrow{\iota} T^n \rightarrow T^n/R \rightarrow 0.$$

Therefore $T^n/R \in \text{Gen}(T) = \text{Pres}(T) = T^\perp$ by Lemma 1.15. This implies that there is an exact sequence

$$(2.2) \quad 0 \rightarrow L \rightarrow T^{(X)} \rightarrow T^n/R \rightarrow 0$$

with $L \in \text{Gen}(T) = T^\perp$. Hence, applying $\text{Hom}_R(-, L)$ to the sequence (2.1), we get

$$\rightarrow \text{Hom}_R(T^n, L) \xrightarrow{\iota^*} \text{Hom}_R(R, L) \rightarrow \text{Ext}^1(T^n/R, L) \rightarrow 0,$$

because $\text{Ext}^1(T, L) = 0$. By Lemma 1.23.(2), ι^* is surjective. This implies that $\text{Ext}^1(T^n/R, L) = 0$. Hence, the sequence (2.2) splits by Proposition 1.8 and we get condition (T1). Thus, T is a tilting module.

(2) \Rightarrow It follows from Proposition 1.21 that T is small and P^\perp is closed under direct sums and epimorphisms. Hence P^\perp is a torsion class. Since P^\perp is closed under epimorphisms and by (T2), $\text{Gen}(P) \subseteq P^\perp$.

\Leftarrow . By Corollary 1.22, P is finitely presented and by Proposition 1.21, $pd(P) \leq 1$. Since $\text{Gen}(P) \subseteq P^\perp$, $\text{Ext}^1(T, T^{(X)}) = 0$ for any set X .

(3) \Rightarrow By (1), $\text{Gen}(T) = T^\perp$. Since T is fin. pres., T is small.

\Leftarrow By (1), T is a tilting module. Since $\text{Gen}(T) = T^\perp$, T^\perp is a torsion class. It follows by Corollary 1.22 that T is finitely presented. \square

Remark 2.3. Note that the proof of (1) of last Proposition shows that every tilting module is faithful and finendo.

The Proposition 2.2, suggest the following generalization of classical partial tilting module.

Definition 2.4. A left R -module P is a *partial tilting module* if $\text{Gen}(P) \subset P^\perp$ and P^\perp is a torsion class.

Remark 2.5.

- Classical tilting module if and only if tilting and small (or just fin. gen.).
- Classical partial tilting module if and only if partial tilting and small (or just fin. gen.).
- Any direct sum of copies of a (partial) tilting module is a (partial) tilting module. (This is something which does not happen in the classical case)

A classical partial tilting module is defined as a finitely presented module satisfying (T2) and (T3). Every partial tilting module P satisfies conditions (T2) and (T3) but next example shows that conditions (T2) and (T3) are not sufficient for P to be partial tilting.

Example 2.6. Let $R = \mathbb{Z}$ and $P = {}_{\mathbb{Z}}\mathbb{Q}$. Since R is a hereditary ring, $pd(P) \leq 1$. Moreover, $\text{Ext}^1(P, P^{(X)}) = 0$ for any set X because R is Noetherian. Thus, P satisfies (T2) and (T3). Note that $\text{Gen}(P)$ consists of all divisible groups. On the other hand $P^\perp = \{G \mid \text{Ext}^1(\mathbb{Q}, G) = 0\}$ is the class of cotorsion groups which is not a torsion class because it is not closed under direct sums. For, consider a prime number p and the abelian group $G = \bigoplus_{n>0} \mathbb{Z}_{p^n}$. It follows from [7, Corollary 54.4] that each \mathbb{Z}_{p^n} is a cotorsion group but G is not. Note that P^\perp is closed under epimorphisms (Lemma 1.17).

Clearly a summand of a classical tilting module is a classical partial tilting module. The converse is not true in general.

Example 2.7. Let k be a universal differential field of characteristic 0 with differentiation D . Denote by $R = k[y; D]$ the ring of differential polynomials of one indeterminate y over k . In [5, Theorem 1.4], it is proved that R is a left and right principal ideal domain. Hence R is Noetherian and hereditary. Moreover R has only one simple left R -module S up to isomorphism which is injective. Under this hypothesis it is not difficult to prove that S is a classical partial tilting R -module. Suppose that there exists a classical tilting module T such that $S \leq^\oplus T$. If T is not injective, $E(T)/T$ is nonzero torsion (singular) R -module. It follows from [2, Theorem 4] that every torsion (singular) module is semisimple. Thus $E(T)/T$ is semisimple and so $E(T)/T \cong S^{(X)}$ for some set X . Therefore there exists a split monomorphism $\phi : S \rightarrow E(T)/T$. Applying $\text{Hom}_R(S, _)$ to the sequence $0 \rightarrow T \rightarrow E(T) \rightarrow E(T)/T \rightarrow 0$ we get

$$\longrightarrow \text{Hom}_R(S, E(T)) \xrightarrow{\pi_*} \text{Hom}_R(S, E(T)/T) \longrightarrow \text{Ext}^1(S, T) \longrightarrow 0$$

where $\pi : E(T) \rightarrow E(T)/T$ is the canonical projection. If there exists $0 \neq \psi \in \text{Hom}_R(S, E(T))$ such that $\phi = \pi_*(\psi) = \pi\psi$, then $S \cong \psi(S) \leq^\oplus E(T)$ and $0 \neq \phi(S) = \pi\psi(S)$. This implies that $\psi(S) \cap T = 0$ and so $\psi(S) = 0$ which is a contradiction. Thus, $\phi \notin \text{Im } \pi_*$. Therefore $\text{Ext}^1(S, T) \neq 0$ which cannot be because T is tilting. Hence T is injective and finitely generated. By [2, Corollary 6], T is semisimple and so $T \cong S^{(X)}$ for some set X . Therefore the condition (T1) implies that R is semisimple, contradiction. Thus, S cannot be a direct summand of a classical tilting R -module. Nevertheless, S is a direct summand of the tilting module $T = S \oplus E(R)$. By [8, Corollary 7.12], $E(R)$ is isomorphic to the skew field of fractions of R . Hence $E(R)$ is a flat non projective R -module. It follows from [11, Theorem 4.30] that $E(R)$ is not finitely generated and so T is not a classical tilting module. In fact T does not satisfy condition (T1₀). For, suppose that there exist $n, m \in \mathbb{N}$ and an exact sequence $0 \rightarrow R \xrightarrow{\nu} T^{(m)} \rightarrow T' \rightarrow 0$ with $T' \leq^\oplus T^{(n)}$. Since T is injective, ν can be extended to a monomorphism $\nu' : E(R) \rightarrow T^{(m)}$. Hence $\nu'(E(R))/\nu(R) \leq T^{(m)}/\nu(R) \cong T'$. Thus T' contains a copy of $E(R)/R$ which is torsion and hence semisimple. Then, $E(R)/R \cong S^{(I)}$ for some infinite set I because $E(R)/R$ is not finitely generated. Therefore,

$$S^{(I)} \hookrightarrow \text{Soc}(T') \leq \text{Soc}(T^{(n)}) = S^{(n)},$$

contradiction. Note that P is also a direct summand of $E(R) \oplus E(R)/R$ which is tilting by the next proposition.

Proposition 2.8. *Let R be a left hereditary left Noetherian ring. Then $T = E(R) \oplus E(R)/R$ is a tilting module.*

Proof. Since R is left hereditary, T is injective and $pd(T) \leq 1$. Moreover, $T^{(X)}$ is injective for any set X . Hence, $\text{Ext}^1(T, T^{(X)}) = 0$ for any set X . We have an exact sequence $0 \rightarrow R \rightarrow E(R) \rightarrow E(R)/R \rightarrow 0$ with $E(R), E(R)/R \in \text{Add}(T)$. Thus, T is a tilting module. \square

Lemma 2.9. *Let P be a module satisfying (T2) and (T3).*

- (1) *Then there is a module T such that P is a summand of T , $\text{Gen}(P) \subseteq P^\perp = T^\perp \subseteq \text{Gen}(T)$, and T satisfies (T1), (T3) and $\text{Ext}^1(T, T) = 0$.*
(2) *Let T be as in (1). Then P is partial tilting if and only if T is tilting.*

Proof. (1) Let $\{\alpha_i\}_{i \in I}$ be a set of generators of $\text{Ext}^1(P, R)$. Each α_i corresponds to an extension of R by P , that is, $0 \rightarrow R \rightarrow M_i \rightarrow P \rightarrow 0$ for all $i \in I$. Taking the direct sum over I of this sequences, we get an exact sequence $0 \rightarrow R^{(I)} \xrightarrow{j} \bigoplus_{i \in I} M_i \rightarrow P^{(I)} \rightarrow 0$. Let $h : R^{(I)} \rightarrow R$ be the homomorphism given by $h((r_i)_{i \in I}) = \sum_{i \in I} r_i$. Taking the push-out of j and h , we get a diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & R & \longrightarrow & M_i & \longrightarrow & P & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & R^{(I)} & \xrightarrow{j} & \bigoplus_{i \in I} M_i & \longrightarrow & P^{(I)} & \longrightarrow & 0 \\
& & \downarrow h & & \downarrow & & \downarrow 1 & & \\
0 & \longrightarrow & R & \longrightarrow & P_0 & \longrightarrow & P^{(I)} & \longrightarrow & 0 \dots \dots (\star)
\end{array}$$

This implies that $P \in \text{Gen}(P_0)$. Applying the functor $\text{Hom}_R(P, -)$ to (\star) , we get the exact sequence

$$\longrightarrow \text{Hom}_R(P, P^{(I)}) \xrightarrow{\partial} \text{Ext}^1(P, R) \longrightarrow \text{Ext}^1(P, P_0) \longrightarrow \text{Ext}^1(P, P^{(I)}).$$

Since P satisfies (T2), $\text{Ext}^1(P, P^{(I)}) = 0$. By construction ∂ is surjective (see Lemma 1.10) and hence $\text{Ext}^1(P, P_0) = 0$. Thus, $P_0 \in P^\perp$. Let $M \in P^\perp$ and we apply $\text{Hom}_R(-, M)$ to (\star) .

$$\rightarrow \text{Hom}_R(P_0, M) \xrightarrow{i^*} \text{Hom}_R(R, M) \rightarrow \text{Ext}^1(P^{(I)}, M) \rightarrow \text{Ext}^1(P_0, M) \rightarrow \text{Ext}^1(R, M)$$

Since $M \in P^\perp$, $\text{Ext}^1(P^{(I)}, M) = 0$ and $\text{Ext}^1(R, M) = 0$ because R is projective. Therefore, i^* is surjective and $\text{Ext}^1(P_0, M) = 0$. This implies that M is P_0 -generated and so $M \in P_0^\perp$. Thus, $P^\perp \subseteq \text{Gen}(P_0) \cap P_0^\perp$. Put $T = P \oplus P_0$. Then $T^\perp = P^\perp \cap P_0^\perp = P^\perp$. Since $P \in \text{Gen}(P_0)$, $\text{Gen}(T) = \text{Gen}(P_0)$. It follows that $T^\perp = P^\perp \subseteq \text{Gen}(P_0) = \text{Gen}(T)$. Also, note that $\text{Ext}^1(P, P) = 0$ by hypothesis, $\text{Ext}^1(P, P_0) = 0$ because $P_0 \in P^\perp$, $\text{Ext}^1(P_0, P) = 0$ because $P^\perp \subseteq \text{Gen}(P_0) \cap P_0^\perp$ and $\text{Ext}^1(P_0, P_0) = 0$ because $P_0 \in P^\perp \subseteq \text{Gen}(P_0) \cap P_0^\perp$. Thus, $\text{Ext}^1(T, T) = 0$. By (\star) , T satisfies (T1). Let M be any module. Applying $\text{Hom}_R(-, M)$ to (\star) , for $n \geq 2$, we get

$$\rightarrow \text{Ext}^{n-1}(R, M) \rightarrow \text{Ext}^n(P^{(I)}, M) \rightarrow \text{Ext}^n(P_0, M) \rightarrow \text{Ext}^n(R, M) \rightarrow .$$

Since R is projective, $\text{Ext}^n(P^{(I)}, M) \cong \text{Ext}^n(P_0, M)$. Since P satisfies (T3), $\text{Ext}^n(P^{(I)}, M) = 0$. This implies that $\text{Ext}^n(P_0, M) = 0$ for all $n \geq 2$ and all module M , that is, $pd(P_0) \leq 1$. Thus, T satisfies (T3).

(2) Let T be as in (1). Suppose P is partial tilting. By hypothesis, $T \in T^\perp = P^\perp$. Therefore $\text{Gen}(T) \subset T^\perp = P^\perp \subseteq \text{Gen}(T)$. By Proposition 2.2, T is a tilting module. Reciprocally, if T is tilting, $P^\perp = T^\perp = \text{Gen}(T)$ by Proposition 2.2. This implies that P^\perp is a torsion class. By hypothesis, $\text{Gen}(P) \subseteq P^\perp$. Thus, P is partial tilting. \square

Theorem 2.10. *Let P be a left R -module. Then, P is a partial tilting module if and only if P is a direct summand of a tilting module T such that $T^\perp = P^\perp$. Moreover, T can be chosen so that $T \cong P \oplus T$.*

Proof. \Rightarrow There is a tilting module T satisfying the conditions in Lemma 2.9. Put $\bar{T} = (T \oplus P)^{(\aleph_0)}$. Since P is a direct summand of T , $\text{Gen}(T) = \text{Gen}(\bar{T})$. Also, $T^\perp = \bar{T}^\perp$ because $P^\perp = T^\perp$. Therefore $\bar{T}^\perp = \text{Gen}(\bar{T})$, that is, \bar{T} is a tilting module. Note that $\bar{T} \cong P \oplus \bar{T}$ and $P^\perp = \bar{T}^\perp$.

\Leftarrow Suppose P is a direct summand of a tilting module T with $T^\perp = P^\perp$. Then $\text{Gen}(P) \subseteq \text{Gen}(T) = T^\perp = P^\perp$. Since $P^\perp = T^\perp = \text{Gen}(T)$, P^\perp is a torsion class. Thus, P is a partial tilting module. \square

We have that, if P is partial tilting module, there exists a module C such that $T = P \oplus C$ is a tilting module and T and P have the same Ext-orthogonal class. Sometimes, C is called *the Bongartz complement of P* . Nevertheless, P can be, at the same time, a direct summand of another tilting module T' with different Ext-orthogonal class

Example 2.11. Let R be the subring of $\text{Mat}_2(\mathbb{C})$ given by $\{(\begin{smallmatrix} a & 0 \\ b & c \end{smallmatrix}) \mid a \in \mathbb{R}, b, c \in \mathbb{C}\}$. Hence R is a finite dimensional hereditary \mathbb{R} -algebra. Consider $P = E(R) = \text{Mat}_2(\mathbb{C})$. By Proposition 2.8, $T' = P \oplus P/R$ is a (classical) tilting module and so P is a (classical) partial tilting module. We have that

$$\mathfrak{I} = \text{Gen}(P) = \text{Gen}(T') = T'^\perp \subseteq P^\perp,$$

where \mathfrak{I} is the class of injective R -modules. We claim that $T'^\perp \neq P^\perp$. Put $N = \{(\begin{smallmatrix} a & 0 \\ b & 0 \end{smallmatrix}) \mid a \in \mathbb{R}, b \in \mathbb{C}\} = R(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix})$ which is a direct summand of R . We have that ${}_R P = \{(\begin{smallmatrix} a & 0 \\ b & 0 \end{smallmatrix}) \mid a, b \in \mathbb{C}\} \oplus \{(\begin{smallmatrix} 0 & d \\ 0 & c \end{smallmatrix}) \mid c, d \in \mathbb{C}\}$ and let P_1 and P_2 denote these direct summands respectively. Hence $P_1 = E(N)$, and $P/R = \frac{P_1 \oplus P_2}{N \oplus N'} \cong \frac{P_1}{N} \oplus \frac{P_2}{N'}$. Note that $P_1 \cong P_2$. We claim that $N \in P^\perp \setminus T'^\perp$, that is $N \in P_1^\perp \setminus \text{Gen}(T')$. Since N is not injective, $N \notin \text{Gen}(T')$. Write $P_1 = Rx + Ry$ with $x = (\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix})$ and $y = (\begin{smallmatrix} i & 0 \\ 0 & 0 \end{smallmatrix})$. Let $\phi \in \text{Hom}_R(P_1, P_1/N)$ given by $\phi(x) = (\begin{smallmatrix} ci & 0 \\ 0 & 0 \end{smallmatrix}) + N$ and $\phi(y) = (\begin{smallmatrix} di & 0 \\ 0 & 0 \end{smallmatrix}) + N$ with $c, d \in \mathbb{R}$. Define $\varphi \in \text{Hom}_R(P_1, P_1)$ as $\varphi(x) = (\begin{smallmatrix} d+ci & 0 \\ 0 & 0 \end{smallmatrix})$ and $\varphi(y) = (\begin{smallmatrix} -c+di & 0 \\ 0 & 0 \end{smallmatrix})$. Then $\phi = \pi\varphi$ where $\pi : P_1 \rightarrow P_1/N$ is the canonical projection. Hence, in the exact sequence

$$\rightarrow \text{Hom}_R(P_1, P_1) \xrightarrow{\pi_*} \text{Hom}_R(P_1, P_1/N) \rightarrow \text{Ext}^1(P_1, N) \rightarrow \text{Ext}^1(P_1, P_1)$$

π_* is surjective and $\text{Ext}^1(P_1, P_1) = 0$ because P_1 is injective. This implies that $\text{Ext}^1(P_1, N) = 0$ and so $N \in P_1^\perp$.

3. TILTING AND CLASSICAL TILTING TORSION THEORIES

Definition 3.1. Let $(\mathcal{T}, \mathcal{F})$ be a (not necessarily hereditary) torsion theory in $R\text{-Mod}$. Then $(\mathcal{T}, \mathcal{F})$ is a (classical) *tilting torsion theory* provided there is a (classical) tilting module T such that $\mathcal{T} = T^\perp$. In this case, \mathcal{T} is called a (classical) *tilting torsion class*.

Recall that if M is a left R -module, the *torsion theory generated by M* is the pair $(\mathcal{T}_M, \mathcal{F}_M)$ where $\mathcal{F}_M = \text{Ker Hom}_R(M, -)$ and $\mathcal{T}_M = \{N \mid \text{Hom}_R(N, F) \forall F \in \mathcal{F}_M\}$. The class \mathcal{T}_M is the least torsion class containing M . It follows that $\text{Gen}(M) \subset \mathcal{T}_M$ for all module M . Now, if $(\mathcal{T}, \mathcal{F})$ is a tilting torsion theory, that is, $\mathcal{T} = T^\perp$ for some tilting module T , then $\text{Gen}(T) = T^\perp = \mathcal{T}$. This implies that $\mathcal{T}_T = \text{Gen}(T) = \mathcal{T}$ and $\mathcal{F} = \mathcal{F}_T$. Thus, $(\mathcal{T}, \mathcal{F})$ is the torsion theory generated by T .

Theorem 3.2. *A torsion class \mathcal{T} in $R\text{-Mod}$ is a tilting torsion class if and only if $\mathcal{T} = P^\perp$ for some $P \in \mathcal{T}$.*

Proof. \Rightarrow If \mathcal{T} is a tilting torsion class, then $\mathcal{T} = T^\perp$ for some tilting module T . Since T is tilting, $T \in \mathcal{T}$.

\Leftarrow Suppose $\mathcal{T} = P^\perp$ for some $P \in \mathcal{T}$. Then, $\text{Gen}(P) \subseteq \mathcal{T} = P^\perp$. Hence P is a partial tilting module. By Theorem 2.10 there exists a tilting module T such that $T^\perp = P^\perp = \mathcal{T}$. Thus, \mathcal{T} is a tilting torsion class. \square

The next result shows that, if P is a (classical) partial tilting module, but not a (classical) tilting, then there are two different torsion theories determined by P .

Lemma 3.3. *If P is a partial tilting module, then both $\text{Gen}(P)$ and P^\perp are torsion classes, the second one being a tilting torsion class.*

Proof. By definition P^\perp is a torsion class. It follows from Theorem 3.2 that P^\perp is a tilting torsion class. It just remains to prove that $\text{Gen}(P)$ is closed under extensions. Let B any module. Consider the exact sequence

$$0 \rightarrow \text{tr}^P(B) \rightarrow B \rightarrow B/\text{tr}^P(B) \rightarrow 0.$$

Applying the functor $\text{Hom}_R(P, -)$ to this sequence, we get

$$\rightarrow \text{Hom}_R(P, B) \rightarrow \text{Hom}_R(P, B/\text{tr}^P(B)) \rightarrow \text{Ext}^1(P, \text{tr}^P(B)).$$

Since $\text{Gen}(P) \subseteq P^\perp$, $\text{Ext}^1(P, \text{tr}^P(B)) = 0$. This implies that for all $f \in \text{Hom}_R(P, B/\text{tr}^P(B))$ the can be lifted to a $\bar{f} \in \text{Hom}_R(P, B)$. But, $\bar{f}(P) \subseteq \text{tr}^P(B)$. Hence $f = 0$. Thus, $\text{tr}^P(B/\text{tr}^P(B)) = 0$. Thus $\text{tr}^P(-)$ is a radical and $\text{Gen}(P)$ is closed under extensions. \square

Proposition 3.4. *Let P be a (fin. gen.) module such that $\text{Gen}(P) \subseteq P^\perp$. Then, $\text{Gen}(P)$ is a (classical) tilting torsion theory if and only if P is faithful and finendo.*

Proof. \Rightarrow If $\text{Gen}(P) = \text{Gen}(T) = T^\perp$ for some (classical) tilting module T , then P is faithful because T is faithful (Remark 2.3). Moreover, by Proposition 1.1, $P^X \in T^\perp = \text{Gen}(P)$. Thus, P is finendo by Lemma 1.14.

\Leftarrow Let P be a (fin. gen.) faithful and finendo module such that $\text{Gen}(P) \subseteq P^\perp$. By Lemma 1.23.(1), there exists an exact sequence $0 \rightarrow R \rightarrow P^n \rightarrow P' \rightarrow 0$. Let $M \in \text{Gen}(P)$. By Lemma 1.23.(3), $P'^\perp \subseteq \text{Gen}(P) =$. On the other hand, if $M \in \text{Gen}(P)$, then $\text{Ext}^1(P^n, M) = 0$. Lemma 1.23.(2) implies that $\text{Ext}^1(P', M) = 0$. Hence $M \in P'^\perp$. Put $T = P \oplus P'$. Then, $\text{Gen}(P) = \text{Gen}(T)$ and $T^\perp = P^\perp \cap P'^\perp = P^\perp \cap \text{Gen}(P) = \text{Gen}(P)$. Hence $\text{Gen}(P) = \text{Gen}(T) = T^\perp$ is a tilting torsion class. Note that, if P is fin. gen. then so is T . \square

Definition 3.5. Let \mathcal{T} be a class of modules A module P is \mathcal{T} -projective if the functor $\text{Hom}_R(P, -)$ preserves exactness of all sequences of the form $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$, where $L, M, N \in \mathcal{T}$.

Remark 3.6. If $\text{Gen}(P) \subseteq P^\perp$, then P is $\text{Gen}(P)$ -projective. Indeed, let $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$ be an exact sequence with $L, M, N \in \text{Gen}(P)$. Applying the functor $\text{Hom}_R(P, -)$ to this sequence, we get

$$0 \rightarrow \text{Hom}_R(P, L) \rightarrow \text{Hom}_R(P, M) \rightarrow \text{Hom}_R(P, N) \rightarrow \text{Ext}^1(P, L)$$

Since $\text{Gen}(P) \subseteq P^\perp$, $\text{Ext}^1(P, L) = 0$. Thus, P is $\text{Gen}(P)$ -projective.

Corollary 3.7. *A class of modules \mathcal{T} is a (classical) tilting torsion class if and only if $\mathcal{T} = \text{Gen}(P)$ for a (fin. gen.) faithful, finendo, and \mathcal{T} -projective module.*

Proof. \Rightarrow Suppose $\mathcal{T} = \text{Gen}(T) = T^\perp$ for some tilting module T . Then, T is faithful and finendo. By last remark, T is \mathcal{T} -projective.

\Leftarrow By Proposition 3.4, it is enough to prove that $\text{Gen}(P) \subseteq P^\perp$. Let $M \in \text{Gen}(P)$. Consider the sequence $0 \rightarrow M \rightarrow E(M) \rightarrow E(M)/M \rightarrow 0$. Note that $M, E(M), E(M)/M \in \text{Gen}(P)$ by Lemma 1.23. Applying the functor $\text{Hom}_R(P, -)$, we get

$$\rightarrow \text{Hom}_R(P, E(M)) \rightarrow \text{Hom}_R(P, E(M)/M) \rightarrow \text{Ext}^1(P, M) \rightarrow \text{Ext}^1(P, E(M)).$$

It follows that $\text{Ext}^1(P, M) = 0$ because P is $\text{Gen}(P)$ -projective and $\text{Ext}^1(P, E(M)) = 0$. Thus, $M \in P^\perp$. \square

Let P be a partial tilting module. Let $[\text{Gen}(P), P^\perp]$ denote the interval of torsion classes \mathcal{T} such that $\text{Gen}(P) \subseteq \mathcal{T} \subseteq P^\perp$. The tilting torsion classes in this interval are characterized as follows.

Lemma 3.8. *Let P be a partial tilting module and let T be any module. The following conditions are equivalent:*

- (a) T is a tilting module and $P \in \text{Add}(T)$;
- (b) $\text{Gen}(T) = T^\perp \in [\text{Gen}(P), P^\perp]$.

Proof. (a) \Rightarrow (b) Since T is tilting, $\text{Gen}(T) = T^\perp$ is a torsion class. Moreover, $\text{Gen}(P) \subseteq \text{Gen}(T)$ and $T^\perp \subseteq P^\perp$ because $P \in \text{Add}(T)$.

(b) \Rightarrow (a) $\text{Gen}(T) = T^\perp$ implies that T is a tilting module. By Lemma 1.15, $P \in \text{Pres}(T)$. Since $\text{Gen}(T) \subseteq P^\perp$, $P \in \text{Add}(T)$ by Lemma 1.16. \square

Proposition 3.9. *Let T_1 and T_2 be two tilting modules. The following conditions are equivalent:*

- (a) $T_1 \in \text{Add}(T_2)$;
- (b) $T_2 \in \text{Add}(T_1)$;
- (c) $T_1 \in T_2^\perp$ and $T_2 \in T_1^\perp$;
- (d) $\text{Gen}(T_1) = \text{Gen}(T_2)$.

Proof. (a) \Rightarrow (d) Since T_2 is tilting and $T_1 \in \text{Add}(T_2)$, by Lemma 3.8 $\text{Gen}(T_1) \subseteq \text{Gen}(T_2) \subseteq T_1^\perp$. This implies that $\text{Gen}(T_1) = \text{Gen}(T_2)$. (b) \Rightarrow (d) is similar.

(d) \Rightarrow (a) and (b) follows from Lemma 3.8.

(c) \Leftrightarrow (d) is clear because $\text{Gen}(T_1) = T_1^\perp$ and $\text{Gen}(T_2) = T_2^\perp$. \square

Example 3.10. Let K be a field. Consider the ring of lower triangular matrices $R = \begin{pmatrix} K & 0 \\ K & K \end{pmatrix}$ with coefficients in K . We have that ${}_R R = \begin{pmatrix} K & 0 \\ K & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 \\ 0 & K \end{pmatrix}$. The injective hull of R is $\text{Mat}_2(K) = \begin{pmatrix} K & 0 \\ K & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & K \\ 0 & K \end{pmatrix}$. There are, up to isomorphism, two simple R -modules. One is $\begin{pmatrix} 0 & 0 \\ 0 & K \end{pmatrix}$ with injective hull $P_1 = \begin{pmatrix} 0 & K \\ 0 & K \end{pmatrix} \cong \begin{pmatrix} K & 0 \\ K & 0 \end{pmatrix}$ and the other one is $P_2 := P_1 / \begin{pmatrix} 0 & 0 \\ 0 & K \end{pmatrix}$ which is injective because R is a left hereditary ring. Since R is left Artinian, every injective R -module is a direct sum of injective hulls of simple

modules, that is, a direct sum of direct sums of copies of P_1 and P_2 . Since P_1 generates P_2 , $\text{Gen}(P_1) = \mathfrak{I}$ the class of all injective modules. On the other hand, $P_1^\perp = R\text{-Mod}$ because P_1 is projective. Hence P_1 is a partial tilting module. Now, let $M \in P_2^\perp$ and let N be any module. Applying $\text{Hom}_R(-, M)$ to the sequence $0 \rightarrow N \rightarrow E(N) \rightarrow E(N)/N \rightarrow 0$, we get

$$\rightarrow \text{Ext}^1(E(N)/N, M) \rightarrow \text{Ext}^1(E(N), M) \rightarrow \text{Ext}^1(N, M) \rightarrow 0$$

We have that $E(N) = P_1^{(X)} \oplus P_2^{(Y)}$ for some sets X and Y . Since $M \in P_2^\perp$ and $P_1^\perp = R\text{-Mod}$, $\text{Ext}^1(E(N), M) = 0$. Hence $\text{Ext}^1(N, M) = 0$. This implies that M is injective. Since always $\mathfrak{I} \subseteq P_2^\perp$, then $\mathfrak{I} = P_2^\perp$. The class $\text{Gen}(P_2)$ consist of all semisimple injective modules. Thus, P_2 is a partial tilting module.

For what follows, we will need some facts on modules of finite length. We place those results here for the convenience of the reader.

Theorem 3.11. *Let M be an indecomposable modulo of finite length. Then, $\text{End}_R(M)$ is a local ring and the noninvertible elements of $\text{End}_R(M)$ are exactly the nilpotent elements.*

Theorem 3.12. *Let $M \neq 0$. If M is Artinian or Noetherian, then there exist indecomposable submodules M_1, \dots, M_n of M such that $M = \bigoplus_{i=1}^n M_i$. Moreover, if M has finite length, $\text{End}_R(M_i)$ is local for every $1 \leq i \leq n$.*

Lemma 3.13. *Let M be a module of finite length. If $\text{Gen}(M)$ is a torsion class, then there exists a direct summand T of M such that $\text{Gen}(M) = \text{Gen}(T) \subseteq T^\perp$.*

Proof. Since M has finite length, $M = M_1 \oplus \dots \oplus M_n$ with M_i indecomposable. Renumbering if needed, there exists $k \leq n$ such that $M_i \in \text{Gen}(M_k \oplus \dots \oplus M_n)$ for all $1 \leq i \leq n$ and $M_i \notin \text{Gen}(\bigoplus\{M_j \mid k \leq j \leq n \text{ and } i \neq j\})$ for all $k \leq i \leq n$. Put $T = M_k \oplus \dots \oplus M_n$, then $\text{Gen}(M) = \text{Gen}(T)$. Let $k \leq \ell \leq n$. Suppose there is $N \in \text{Gen}(M)$ with $\text{Ext}^1(M_\ell, N) \neq 0$. Set $B = \bigoplus\{M_i \mid k \leq i \leq n \text{ and } i \neq \ell\}$. Then $T = M_\ell \oplus B$. Since $\text{Ext}^1(M_\ell, N) \neq 0$, there is a nontrivial extension $0 \rightarrow N \rightarrow E \rightarrow M_\ell \rightarrow 0$. Hence $E \in \text{Gen}(M) = \text{Gen}(T)$ because $\text{Gen}(M)$ is a torsion class. There is a commutative diagram

$$\begin{array}{ccccccc} & & & T^{(I)} & \xlongequal{\quad} & M_\ell^{(I)} \oplus B^{(I)} & \\ & & & \downarrow \rho & & \downarrow ((f_i)_I, g) & \\ 0 & \longrightarrow & N & \longrightarrow & E & \xrightarrow{\nu} & M_\ell \longrightarrow 0 \end{array}$$

Here $f_i = \nu \rho \eta_i$, where $\eta_i : M_\ell \rightarrow M_\ell^{(I)}$ is the canonical inclusion. Then $f_i \in \text{End}_R(M_\ell)$ is not an isomorphism for all $i \in I$, because ν does not split. Since $\text{End}_R(M_\ell)$ is local, $f_i \in \text{Rad}(\text{End}_R(M_\ell))$ for all $i \in I$. This implies that $\sum_{i \in I} f_i(M_\ell) \subseteq \text{Rad}(\text{End}_R(M_\ell))M_\ell$. Note that $\text{Rad}(\text{End}_R(M_\ell))$ is nilpotent [10, Ex. 21.24], and $M_\ell = \sum_{i \in I} f_i(M_\ell) + g(B^{(I)})$. This implies that $M_\ell = g(B^{(I)}) \in \text{Gen}(B)$ by [10, Proposition 23.16], which is a contradiction. Therefore, $\text{Gen}(M) = \text{Gen}(T)$ and

$$\text{Gen}(M) \subseteq \bigcap_{k \leq \ell \leq n} M_\ell = T^\perp.$$

□

The next example shows that last lemma cannot be true if the module M has infinite length.

Example 3.14. Let $p \in \mathbb{Z}$ be a prime number. Consider $R = \mathbb{Z}$ and $M = \bigoplus_{n>0} \mathbb{Z}_{p^n}$. Let \mathcal{T}_p be the class of p -groups. Then $\text{Gen}(M) = \mathcal{T}_p$ which is a torsion class. Let $0 \neq T \in \mathcal{T}_p$ be any p -group. It follows that $E(T) \cong \mathbb{Z}_{p^\infty}^{(X)}$ for some set X . Then,

$$\frac{\mathbb{Z}_{p^\infty}^{(X)}}{\mathbb{Z}_p^{(X)}} \cong \left(\frac{\mathbb{Z}_{p^\infty}}{\mathbb{Z}_p} \right)^{(X)} \cong \mathbb{Z}_{p^\infty}^{(X)} \cong E(T).$$

Therefore, there is a monomorphism $\alpha : T \rightarrow \frac{\mathbb{Z}_{p^\infty}^{(X)}}{\mathbb{Z}_p^{(X)}}$. Consider the following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z}_p^{(X)} & \longrightarrow & A & \longrightarrow & T & \longrightarrow & 0 \\ & & \downarrow 1 & & \downarrow & & \downarrow \alpha & & \\ 0 & \longrightarrow & \mathbb{Z}_p^{(X)} & \xrightarrow{i} & \mathbb{Z}_{p^\infty}^{(X)} & \xrightarrow{\pi} & \frac{\mathbb{Z}_{p^\infty}^{(X)}}{\mathbb{Z}_p^{(X)}} & \longrightarrow & 0 \end{array}$$

where the lower row is the canonical sequence and A is the pull-back of α and π . By [13, Lemma 7.29], the upper row is exact. Since i is an essential monomorphism, the upper row is not a trivial extension. This implies that $\text{Ext}^1(T, \mathbb{Z}_p^{(X)}) \neq 0$. Thus, $\text{Gen}(T) \not\subseteq T^\perp$.

Theorem 3.15. *Consider the following conditions for a torsion class \mathcal{T} in $R\text{-Mod}$.*

- (1) \mathcal{T} is a classic tilting torsion class.
- (2) \mathcal{T} is closed under direct products, it contains any injective module and $\mathcal{T} = \text{Gen}(P)$ for a finitely generated module P .
- (3) $\mathcal{T} = \text{Gen}(P)$ for a finitely generated, faithful and finendo module P .

Then, (1) \Rightarrow (2) \Rightarrow (3). In addition, if R is left Artinian, then the three conditions are equivalent.

Proof. (1) \Rightarrow (2) Suppose $\mathcal{T} = \text{Gen}(P) = P^\perp$ is a classical torsion class. Since $\mathcal{T} = P^\perp$, \mathcal{T} is closed under direct products (Proposition 1.1) and contains any injective module. By hypothesis, P is finitely generated.

(2) \Rightarrow (3) By hypothesis, $P^X \in \text{Gen}(P) = \mathcal{T}$ for every set X . This implies that P is finendo (Lemma 1.14). Since $E(R) \in \mathcal{T} = \text{Gen}(P)$, there exists an epimorphism $P^{(X)} \rightarrow E(R) \rightarrow 0$ for some set X . Since R is projective, the inclusion $R \hookrightarrow E(R)$ lifts to a monomorphism $R \rightarrow P^{(X)}$. Then, P is faithful.

Now suppose R is left Artinian and assume (3). Then P is of finite length. By Lemma 3.13, there is a direct summand T of P such that $\mathcal{T} = \text{Gen}(P) = \text{Gen}(T) \subseteq T^\perp$. Since P is faithful and there exists an epimorphism $T^{(X)} \rightarrow P$ for some set X , T is also faithful. Now, for any set X , $P^X \in \text{Gen}(P)$ because P is finendo. This implies that $T^X \in \text{Gen}(P) = \text{Gen}(T)$. Thus, T is finendo. By Proposition 3.4, \mathcal{T} is a classical tilting torsion class, proving (1). \square

Remark 3.16. Note that either (1), (2) or (3) of Theorem 3.15 does not imply that P is of finite length. For, just consider a $P = R$ for some non left Artinian ring R .

Definition 3.17. A bimodule ${}_A C_B$ is faithfully balanced if the natural homomorphism $A \rightarrow \text{End}_B(C)$ and $B \rightarrow \text{End}_A(C)$ are isomorphisms.

For a nonclassical torsion class $\mathcal{T} = \text{Gen}(T)$ there is not a generalization of the Brenner-Butler Theorem, that is, there is not an equivalence of categories between \mathcal{T} and $\text{Cogen}(\text{Hom}_R(-, T)) = \text{Ker}(\text{Tor}_1^S(-, T))$. This is because T is not finitely generated. What can be done is to choose T as a classical partial tilting faithfully balanced module over its endomorphism ring such that \mathcal{T} is equivalent to $\text{Hom}_R(T, \mathcal{T})$.

Lemma 3.18. *Let T be an R -module with endomorphism ring $S = \text{End}_R(T)$. Consider the following conditions:*

(1) T satisfies:

(T1₀) *There is an exact sequence $0 \rightarrow R \rightarrow T' \rightarrow T'' \rightarrow 0$ such that $T', T'' \in \text{add}(T)$.*

(T2₀) $\text{Ext}_R^1(T, T) = 0$.

(2) T is faithfully balanced as $S - R$ -bimodule and ${}_S T$ is a classical partial tilting module.

(3) ${}_R T$ is faithful and there is $\bar{t} = (t_1, \dots, t_n) \in T^n$ such that ${}_S \langle t_1, \dots, t_n \rangle = {}_S T$ and $T^n / R\bar{t} \in \text{add}(T)$.

(4) ${}_R T$ satisfies (T1₀).

Then (1) \Rightarrow (2) \Leftrightarrow (3) \Rightarrow (4). Moreover, if (3) is true, then every module $M \in \text{Gen}(T)$ is T -reflexive, i.e., $\text{Hom}_R(T, M) \otimes_S T \cong M$ canonically.

Proof. (1) \Rightarrow (2) Applying the functor $\text{Hom}_R(-, T)$ to the sequence (T1₀), we get a sequence in $S\text{-Mod}$:

$$(3.1) \quad 0 \rightarrow \text{Hom}_R(T'', T) \rightarrow \text{Hom}_R(T', T) \rightarrow \text{Hom}_R(R, T) \rightarrow \text{Ext}_R^1(T'', T) = 0$$

where $\text{Ext}_R^1(T'', T) = 0$ because (T2₀). Now, we have that $T' \leq^{\oplus} T^m$ for some $m > 0$. Then $\text{Hom}_R(T', T) \leq^{\oplus} \text{Hom}_R(T^m, T) \cong S^m$. Therefore, $\text{Hom}_R(T', T), \text{Hom}_R(T'', T) \in \text{add}(S)$, that is, $\text{Hom}_R(T', T), \text{Hom}_R(T'', T)$ are finitely generated projective S -modules. Since ${}_S \text{Hom}_R(R, T) \cong {}_S T$, ${}_S T$ satisfies (T3) and (T4). Now, we apply $\text{Hom}_S(-, T)$ to (3.1) and we get the diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & R & \longrightarrow & T' & \longrightarrow & T'' \longrightarrow 0 \\ & & \omega_R \downarrow & & \omega_{T'} \downarrow & & \omega_{T''} \downarrow \\ 0 & \longrightarrow & \text{Hom}_S(\text{Hom}_R(R, T), T) & \longrightarrow & \text{Hom}_S(\text{Hom}_R(T', T), T) & \longrightarrow & \text{Hom}_S(\text{Hom}_R(T'', T), T) \longrightarrow \text{Ext}_S^1(T, T) \end{array}$$

Note that $\text{Hom}_S(\text{Hom}_R(R, T), T) \cong \text{End}_S(T)$ and $\omega_{T'}$ and $\omega_{T''}$ are isomorphisms because $T', T'' \in \text{add}(T)$. Thus, ω_R is an isomorphism. This implies that T is faithfully balanced. Also, we have that $\text{Ext}_S^1(T, T) = 0$.

(2) \Rightarrow (3) Since ${}_S T$ satisfies (T3) and (T4), there is an exact sequence in $S\text{-Mod}$

$$0 \longrightarrow K \longrightarrow S^n \xrightarrow{\phi} T \longrightarrow 0$$

with $K \in \text{add}(S)$. Let $\{e_i\}$ be the canonical basis of S^n . Then, ${}_S T = {}_S \langle t_1, \dots, t_n \rangle$ where $t_i = \phi(e_i)$ for $1 \leq i \leq n$. Applying the functor $\text{Hom}_S(-, T)$ to the sequence, we get

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{Hom}_S(T, T) & \xrightarrow{\phi^*} & \text{Hom}_S(S^n, T) & \longrightarrow & \text{Hom}_S(K, T) \longrightarrow \text{Ext}_S^1(T, T) = 0 \\ & & \cong \downarrow & & \cong \downarrow & & \downarrow \\ 0 & \longrightarrow & R & \xrightarrow{\phi^*} & T^n & \longrightarrow & T^n / R\bar{t} \longrightarrow 0 \end{array}$$

where $\bar{t} = \phi^*(1) = (t_1, \dots, t_n)$. The first isomorphism is by hypothesis and the second is the canonical isomorphism. Hence $T^n/R\bar{t} \cong \text{Hom}_S(K, T) \in \text{add}(T)$.

(3) \Rightarrow (2) Since ${}_R T$ is faithful and ${}_S T = {}_S \langle t_1, \dots, t_n \rangle$, $R\bar{t} = R$. Hence, there is an exact sequence $0 \longrightarrow R \xrightarrow{i} T^n \longrightarrow T_0 \longrightarrow 0$, with $T_0 \cong T^n/R\bar{t}$. Applying the functor $\text{Hom}_R(-, T)$ to the sequence, we get a sequence in $S\text{-Mod}$:

$$0 \longrightarrow \text{Hom}_R(T_0, T) \longrightarrow \text{Hom}_R(T^n, T) \xrightarrow{i^*} \text{Hom}_R(R, T) \cong_S T.$$

Given $t \in T$, there exist $f_1, \dots, f_n \in S$ such that $t = \sum_{i=1}^n f_i(t_i)$. Then $i^*(\sum_{i=1}^n f_i)(1) = t$. Thus, i^* is surjective. Also, $T_0 \in \text{add}(S)$. Therefore, ${}_S T$ satisfies (T3) and (T4). Now, we apply $\text{Hom}_S(-, T)$ and we get a commutative diagram in $R\text{-Mod}$,

$$\begin{array}{ccccccc} 0 & \longrightarrow & R & \longrightarrow & T^n & \longrightarrow & T_0 & \longrightarrow & 0 \\ & & \downarrow \omega_R & & \downarrow \cong \omega_{T^n} & & \downarrow \cong \omega_{T_0} & & \\ 0 & \longrightarrow & \text{End}_S(T) & \longrightarrow & \text{Hom}_S(S^n, T) & \longrightarrow & \text{Hom}_S(\text{Hom}_R(T_0, T), T) & \longrightarrow & \text{Ext}^1(T, T), \end{array}$$

where ω_{T^n} and ω_{T_0} are isomorphisms. Thus, ω_R is an isomorphism. Hence, T is faithfully balanced and $\text{Ext}_S^1(T, T) = 0$. At the beginning we show (3) \Rightarrow (4).

For the last assertion, assume (3) and let $M \in \text{Gen}(T)$ and $\rho_M : \text{Hom}_R(T, M) \otimes_S T \rightarrow M$ be the canonical homomorphism given by $\rho_M(\phi \otimes t) = \phi(t)$. Since M is T -generated, each element $m \in M$ can be writing as a finite sum $m = \sum f_i(t_i)$ with $f_i : T \rightarrow M$. Así $\rho_M(\sum f_i \otimes t_i) = m$, that is, ρ_M is surjective. Now, let us prove that ρ_M is injective. Given any element $\sum \phi \otimes t \in \text{Hom}_R(T, M) \otimes_S T$, then

$$\phi \otimes t = \phi \otimes \sum \phi_i(t_i) = \sum \phi \phi_i \otimes t_i = \sum \psi_i \otimes t_i.$$

Hence, if $u \in \text{Ker } \rho_M$, then we can write $u = \sum_{i=1}^n \phi_i \otimes t_i$, $\phi = (\phi_1, \dots, \phi_n) \in \text{Hom}_R(T^n, M)$ with $\phi(\bar{t}) = 0$. Consider the following diagram:

$$\begin{array}{ccc} T^n & \xrightarrow{\phi} & M \\ \pi \downarrow & \nearrow \bar{\phi} & \uparrow \hat{\phi} \\ T^n/R\bar{t} & \xrightarrow[\oplus]{i} & T^m, \end{array}$$

where π is the canonical projection. Since $\phi(\bar{t}) = 0$, then ϕ factors through $T^n/R\bar{t}$. Since $T^n/R\bar{t} \in \text{add}(T)$, there exists $m > 0$ such that $T^n/R\bar{t} \leq^{\oplus} T^m$ and so there exists a homomorphism $\hat{\phi} : T^m \rightarrow M$ such that $\hat{\phi}i = \bar{\phi}$. Therefore, $\phi = \bar{\phi}\pi = \hat{\phi}i\pi = \hat{\phi}(s_{ji})$ where (s_{ji}) a matrix of $m \times n$ with $s_{ji} \in S$. Let $\eta_i : T \rightarrow T^n$ be the

canonical inclusion. Hence

$$\begin{aligned}
u &= \sum_{i=1}^n \phi_i \otimes t_i \\
&= \sum_{i=1}^n \left(\sum_{j=1}^m \eta_j \hat{\phi} s_{ji} \right) \otimes t_i \\
&= \sum_{j=1}^m \eta_j \hat{\phi} \otimes \left(\sum_{i=1}^n s_{ji} t_i \right) \\
&= \sum_{j=1}^m \eta_j \hat{\phi} \otimes (i\pi(\bar{t}))_j \\
&= 0
\end{aligned}$$

Thus, ρ_M is injective. \square

The following examples show that the implications (3) \Rightarrow (1) and (4) \Rightarrow (3) are not true in general.

Example 3.19. Let K be a field.

- (i) Consider the ring $R = \begin{pmatrix} K & 0 \\ K^{(\mathbb{N})} & K \end{pmatrix}$ and the idempotents $\epsilon_a = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $\epsilon_b = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. Then,

$$R = R\epsilon_a \oplus R\epsilon_b = \begin{pmatrix} K & 0 \\ K^{(\mathbb{N})} & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 \\ 0 & K \end{pmatrix}.$$

Let e_1 be the first element in the canonical basis of $K^{(\mathbb{N})}$. Put $S = R \begin{pmatrix} 0 & 0 \\ e_1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ K e_1 & 0 \end{pmatrix}$. Then $S \cong R\epsilon_b$. Set $M_1 = R\epsilon_a$, $M_2 = R\epsilon_a/S$ and $M = M_1 \oplus M_2$. Then, there is an exact sequence:

$$0 \rightarrow M_1 \oplus R\epsilon_b = R \rightarrow M_1^2 \rightarrow M \rightarrow 0.$$

This implies that M satisfies the condition (4) of Lemma 3.18 and M is finitely presented, since M_1 is a finitely generated projective R -module. On the other hand,

$$\text{End}_R(M) = \begin{pmatrix} \text{End}_R(M_1) & \text{Hom}_R(M_2, M_1) \\ \text{Hom}_R(M_1, M_2) & \text{End}_R(M_2) \end{pmatrix} = \begin{pmatrix} K & 0 \\ K & \text{End}_R(M_2) \end{pmatrix}.$$

Since the dimension over K of M_1 is infinite, M cannot be finitely generated over its endomorphism ring. Thus, M does not satisfy the condition (3) of Lemma 3.18.

- (ii) Let R be the ring of 3×3 lower triangular matrices with coefficients in K . Then $R = R\epsilon_1 \oplus R\epsilon_2 \oplus R\epsilon_3$ where

$$\epsilon_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \epsilon_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \epsilon_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then $\text{Rad}(R\epsilon_1) = \begin{pmatrix} 0 & 0 & 0 \\ K & 0 & 0 \\ K & 0 & 0 \end{pmatrix}$ and $\text{Soc}(R\epsilon_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ K & 0 & 0 \end{pmatrix}$. Set $U = R\epsilon_1/\text{Soc}(R\epsilon_1)$, $T = R\epsilon_1 \oplus R\epsilon_3 \oplus R\epsilon_1/\text{Rad}(R\epsilon_1)$ and $M = T \oplus U$. Then T is a tilting module, $\dim_K(U) < \infty$ and $U \in \text{Gen}(T)$. It follows from [6, Proposition 8] that M satisfies the condition (3) of Lemma 3.18. There is a canonical, non trivial,

extension $0 \rightarrow R\epsilon_3 \rightarrow R\epsilon_1 \rightarrow U \rightarrow 0$, because $\text{Soc}(R\epsilon_1) \cong R\epsilon_3$. This implies that there is a non trivial extension of T by U . Thus, $\text{Ext}^1(U, T) \neq 0$ and so $\text{Ext}^1(M, M) \neq 0$. That is, M does not satisfy the condition (1) of Lemma 3.18.

Proposition 3.20. *If T is a tilting module, then there exists a cardinal κ such that the tilting module $T^{(\kappa)}$ satisfies (T1₀).*

Proof. By (T1), there exists an exact sequence $0 \rightarrow R \rightarrow T' \rightarrow T'' \rightarrow 0$ such that $T', T'' \in \text{Add}(T)$. Then, there are two cardinals κ_1 and κ_2 such that $T' \leq^{\oplus} T^{(\kappa_1)}$ and $T'' \leq^{\oplus} T^{(\kappa_2)}$. Take $\kappa = \max\{\kappa_1, \kappa_2\}$. Thus, $T', T'' \in \text{add}(T^{(\kappa)})$. \square

Corollary 3.21. *Let \mathcal{T} be a tilting torsion class in $R\text{-Mod}$. Then \mathcal{T} is generated by a tilting module T such that:*

- (1) *if $S = \text{End}_R(T)$ then T is a faithfully balanced $(S - R)$ -bimodule and ${}_S T$ is a classical partial tilting module.*
- (2) *\mathcal{T} coincides with the class of T -reflexive R -modules, i.e.,*

$$\text{Hom}_R(T, \mathcal{T}) \begin{array}{c} \xrightarrow{-\otimes_S T} \\ \xleftarrow{\text{Hom}_R(T, -)} \end{array} \mathcal{T}$$

is an equivalence.

- (3) *$\text{Hom}_R(T, \mathcal{T})$ is a torsionfree class in $S\text{-Mod}$ if and only if ${}_R T$ is classical tilting.*

Proof. Since $\text{Gen}(T) = \text{Gen}(T^{(\kappa)})$ for any cardinal κ , by Proposition 3.20, we can assume that $\mathcal{T} = \text{Gen}(T)$ for a tilting module T satisfying (T1₀) and (T2₀).

(1) It follows from the condition (2) of Lemma 3.18.

(2) By the condition (3) of Lemma 3.18, every $M \in \text{Gen}(T)$ is T -reflexive. On the other hand, any T reflexive module is T -generated.

(3) \Rightarrow If $\text{Hom}_R(T, \mathcal{T})$ is a torsionfree class in $S\text{-Mod}$, then $\text{Hom}_R(T, \mathcal{T}) = \text{Cogen}(S)$. The equivalence

$$\text{Cogen}(S) \begin{array}{c} \xrightarrow{-\otimes_S T} \\ \xleftarrow{\text{Hom}_R(T, -)} \end{array} \mathcal{T}$$

implies that ${}_R T$ is finitely generated by [15, Theorem 1].

\Leftarrow Since ${}_R T$ is finitely generated and we have the equivalence

$$\text{Hom}_R(T, \mathcal{T}) \begin{array}{c} \xrightarrow{-\otimes_S T} \\ \xleftarrow{\text{Hom}_R(T, -)} \end{array} \mathcal{T}$$

by [12, Theorem 3.1], $\text{Hom}_R(T, \mathcal{T}) = \text{Cogen}(S)$. Thus, $\text{Hom}_R(T, \mathcal{T})$ is a torsionfree class. \square

4. EXERCISES

- (1) Prove Corollary 1.7.
- (2) [13, Ex. 7.26(ii)].
- (3) Let M be a module. Prove that M^\perp is closed under extensions and contains all injective modules.

- (4) Let M be a module. Prove that $\text{Gen}(M)$ is closed under epimorphisms and direct sums.
- (5) Prove Remark 1.20 and give an example of a no finitely generated small module.
- (6) A module T is classical tilting if and only if T satisfies (T1₀), (T2₀), (T3) and (T4).
- (7) A module T is classical partial tilting if and only if T satisfies (T2₀), (T3) and (T4). [9, III.6]
- (8) Prove Remark 2.5.
- (9) Prove that every simple module over a left hereditary left Noetherian left V-ring is a classical partial tilting module.
- (10) Let R be a ring and M be a left R -module. The *singular submodule* of M is defined as $\mathcal{Z}(M) = \{m \in M \mid \text{ann}(m) \leq^{\text{ess}} R\}$. It is said that a module is singular if $\mathcal{Z}(M) = M$. Show that,
- $\mathcal{Z}(M/N) = M/N$ for all $N \leq^{\text{ess}} M$.
 - if R is a semiprime Noetherian ring $\mathcal{Z}(M)$ is equal to the *torsion* of M , that is $\mathcal{Z}(M) = t(M) = \{m \in M \mid cm = 0 \text{ for some regular element } c \in R\}$. [8, Ch. 7]
- (11) Let R be a hereditary Noetherian V-ring. Using [2, Theorem 4] prove that every torsion R -module is semisimple.
- (12) In Example 2.7
- Prove that $T = S \oplus E(R)$ is a tilting module.
 - Find a reference for the sentence “ $E(R)$ is a flat non projective R -module”.
 - Prove that $E(R)/R$ cannot be finitely generated.
- (13) Let E be an injective module and $\varphi : M \rightarrow E$ be a monomorphism. Show that if $\alpha : M \rightarrow N$ is an essential monomorphism, then there exists a monomorphism $\bar{\alpha} : N \rightarrow E$ such that $\bar{\alpha}\alpha = \varphi$.
- (14) In Example 2.11, prove that:
- R is finite dimensional over \mathbb{R} .
 - Describe the lattice of left ideals of the ring R . [10, Proposition 1.7]
 - Prove that R is a hereditary ring. (*Hint*: Prove that all the minimal ideals of R are isomorphic)
 - the left ideals $I = \begin{pmatrix} \mathbb{R} & 0 \\ \mathbb{C} & 0 \end{pmatrix}$ and $J = \begin{pmatrix} 0 & 0 \\ \mathbb{C} & \mathbb{C} \end{pmatrix}$ are two-sided ideals and are the only two maximal ideals of R . Conclude that there are only two isomorphism classes of simple R -modules
 - R is Artinian (use [10, Theorem 1.22])
 - there is an isomorphism:

$$R/J \cong P / \begin{pmatrix} \mathbb{R} & \mathbb{C} \\ \mathbb{C} & \mathbb{C} \end{pmatrix}$$

and hence R/J is injective.

- $\mathfrak{J} = \text{Gen}(P)$. (*Hint*: prove that P generates the injective hull of each simple)
- (15) Let r be a preradical, i.e., a subfunctor of the identity functor. Show that if r is a radical, that is, $r(M/r(M)) = 0$ for all module M , then the class $\mathcal{T}_r = \{M \mid r(M) = M\}$ is closed under extensions.
- (16) in Example 3.10.
- Describe the lattice of left ideals of the ring R . [10, Proposition 1.7]

- (b) Prove that R is an Artinian hereditary ring. (*Hint*: Prove that all the minimal ideals of R are isomorphic)
- (17) [10, Ex. 21.24].
- (18) In Example 3.14, prove the equality $\text{Gen}(M) = \mathcal{T}_p$.
- (19) Prove that the homomorphisms $\omega_{T'}$ and $\omega_{T''}$ in the proof (1) \Rightarrow (2) of Lemma 3.18, are isomorphisms.
- (20) Prove that the module $\text{Hom}_R(M_1, M_2) = K$ and $\text{Hom}_R(M_2, M_1) = 0$ in Example 3.19(i).
- (21) Prove that the module T in Example 3.19(ii) is a tilting module.
- (22) In the proof (3) \Rightarrow , prove that $\text{Hom}_R(T, \mathcal{T}) = \text{Cogen}(S)$.
- (23) In the proof (3) \Leftarrow , prove that $\text{Cogen}(S)$ is a torsionfree class.

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FACULTAD DE CIENCIAS FÍSICO-MATEMÁTICAS, BENEMÉRITA UNIVERSIDAD AUTÓNOMA DE PUEBLA,
 AV. SAN CLAUDIO Y 18 SUR, COL.SAN MANUEL, CIUDAD UNIVERSITARIA, 72570, PUEBLA, MÉXICO
 Email address: mmedina@fcfm.buap.mx